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# Otter's Method and the Homology of Homeomorphically Irreducible $k$-Trees

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Let $\mathscr{T}_n^{(k)}$ denote the collection of trees with $nk+2$ labelled leaves which have the property that every internal node has degree $mk+2$ for some $m \geqslant 1$. Partially order



action of $S_{nk+2}$. We show that the top homology of $\mathscr{T}_n^{(k)}$ as an $S_{nk+2}$-module is

$$(ind_{S_{nk+1}}^{S_{nk+2}}(Lie_{nk+1}^{(k)}))/Lie_{nk+2}^{(k)}$$

where $Lie_N^{(k)}$ is the action of $S_N$ on the $1^N$-homogeneous piece of the free Lie $k$-algebra. This generalizes the result obtained by Sarah Whitehouse [W] in the case $k=1$. 

## 1. INTRODUCTION

Fix a positive integer $k$. For each $n \geqslant 0$ let $\mathscr{T}_n^{(k)}$ denote the collection of all trees having $nk+2$ labelled leaves and having the property that every internal vertex has degree $mk+2$ for some $m>0$. To clarify, $T_0^{(k)}$ contains the single tree


which vacuously satisfies the condition on internal nodes. Note that $\mathscr{T}_n^{(1)}$ consists of the collection of homeomorphically irreducible trees with $n+2$ labelled leaves. We call the elements of $\mathscr{T}_n^{(k)}$ *homeomorphically irreducible $k$-trees.*

Define a partial ordering on the set $\mathscr{T}_n^{(k)}$ by saying that $\tau_1 \leqslant \tau_2$ if $\tau_1$ can be obtained from $\tau_2$ by contracting some set of internal edges. Note that there is a unique minimal element $\hat{0}$ in $\mathscr{T}_n^{(k)}$, this being the star with $nk+2$

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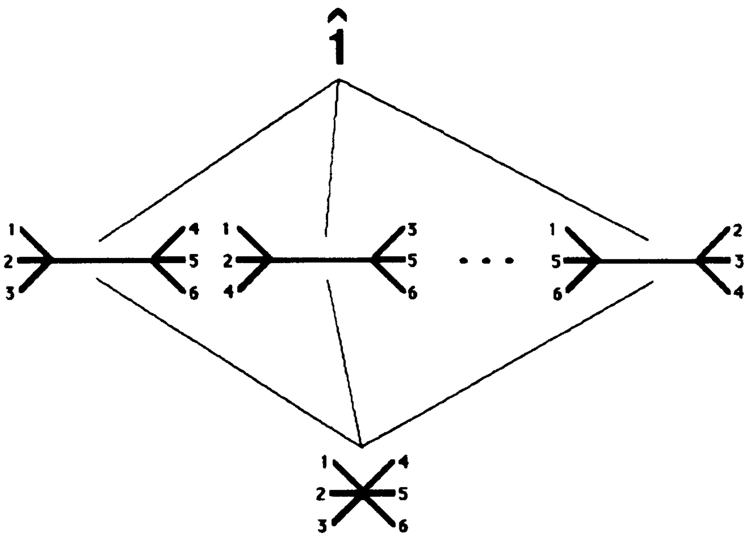
leaves. Also note that contraction of an internal edge preserves the condition that every internal vertex has degree of the form $mk+2$. So for every tree $\tau$ in $\mathcal{T}^{(k)}$ the interval $[\hat{0}, \tau]$ is isomorphic to the lattice of subsets of the internal edges of $\tau$. In particular $\mathcal{T}_n^{(k)}$ is ranked with rank function $\rho(\tau)$ being the number of internal edges of $\tau$.

EXAMPLE. If $n=k=2$, $nk+2=6$. Every internal node must have degree 4 or 6.




Let $\mathcal{L}_n^{(k)}$ denote $\mathcal{T}_n^{(k)}$ together with a new maximal element $\hat{1}$. It is easy to check that $\hat{1}$ has rank $n$ in $\mathcal{L}_n^{(k)}$ for $n \geqslant 1$. In $\mathcal{L}_0^{(k)}$, $\hat{1}$ has rank 1. The group $S_{nk+2}$ acts on the poset $\mathcal{L}_n^{(k)}$ hence on its homology. We will show that $\mathcal{L}_n^{(k)}$ is Cohen–Macaulay hence its homology vanishes except at dimension $n-1$. The main result of this paper identifies $H_{n-1}(\mathcal{L}_n^{(k)})$ as an $S_{nk+2}$-module.

In a recent paper, Hanlon and Wachs [HW] introduce the notion of a "Lie $k$-Algebra" which is analogous to a Lie algebra but with a $(k+1)$-ary operation. The main result of that paper identifies the character $Lie_N^{(k)}$ of $S_N$ acting on the $1^N$ homogeneous piece of the free Lie $k$-algebra with $N$ odd generators. This theorem states that $Lie_N^{(k)}$ is isomorphic, as an $S_N$-module, to the top homology of the poset of partitions of $N$ with every block size congruent to $1 \bmod k$. These latter modules were studied in [CHR] because of the remarkable numerology of their character values.

The main result of this paper gives a description of the homology character $H_{n-1}(\mathcal{L}_n^{(k)})$ in terms of the characters $Lie_n^{(k)}$.

THEOREM 1.1.

$$H_{n-1}(\mathscr{L}_n^{(k)}) \cong (ind_{S_{nk+1}}^{S_{nk+2}}(Lie_{nk+1}^{(k)}))/(Lie_{nk+2}^{(k)}).$$

In this result, the isomorphism denotes an isomorphism of $S_{nk+2}$ modules. The case $k=1$ of Theorem 1.1 was proved by Sarah Whitehouse [W] and related results in the $k=1$ case were proved by Alan Robinson [AR].

The proof that we give for Theorem 1.1 contains an unusual mix of techniques from Algebraic and Enumerative Combinatorics. We will compute the cycle index (i.e., generating function) for the values of the character on the left-hand side. Our first step will be to show that it is the generating function for all pairs $(\tau, \sigma)$ where $\tau$ is an unlabelled tree in $\mathscr{T}_n^{(k)}$ and $\sigma$ is an automorphism of $\tau$. But the pair $(\tau, \sigma)$ must be counted with a sign $(-1)^{E(\tau,\sigma)}$ where $E(\tau, \sigma)$ is the number of cycles of $\sigma$ acting on the edge set of $\tau$. Then we will use Otter's Dissimilarity Characteristic method to evaluate the generating function for these pairs $(\tau, \sigma)$.

I thank Curtis Greene and Jean-Louis Loday for many helpful conversations concerning this work.

## 2. BACKGROUND

The statement and proof of our main result touches on a variety of topics. In this section we review these topics in order to make basic definitions and to establish terminology and notation.

### 2A. *Poset Homology*

Let $P$ be a ranked, finite poset with unique minimal and maximal elements $\hat{0}$ and $\hat{1}$. For each $r$ between 0 and $rk(\hat{1})-1$, let $C_r$ denote the complex vector space with basis consisting of the set of $\hat{0}-\hat{1}$ chains of length $r+1$, i.e., chains of the form $\hat{0}<x_1<x_2<\cdots<x_r<\hat{1}$. Define a linear map $\partial_r\colon C_r \to C_{r-1}$ by

$$\partial_r(\hat{0}<x_1<\cdots<x_r<\hat{1})$$
$$= \sum_{i=1}^{r} (-1)^{i-1} (\hat{0}<x_1<\cdots<x_{i-1}<x_{i+1}<\cdots<x_r<\hat{1}).$$

It is easy to check that $\partial_r \cdot \partial_{r+1}=0$ and so the image of $\partial_{r+1}$ is contained in the kernel of $\partial_r$. Define the $r$ th homology of $P$, $H_r(P)$, to be

$$H_r(P) = \ker \partial_r / im\, \partial_{r+1}.$$

Let $I(P)$ denote the collection of intervals of $P$. Recall that the *Möbius function* $\mu$ of $P$ (see [Ro]) is a function with domain $I(P)$ defined recursively on the size of the interval by

$$\mu(x, y) = \begin{cases} 1 & \text{if } \; x = y \\ -\sum_{x \leqslant z < y} \mu(x, z) & \text{if } \; x < y \end{cases}$$

A classical result which will be important in our analysis is that the Euler characteristic for the homology of $P$ is equal to the value of the Möbius function on the interval $[\hat{0}, \hat{1}]$.

THEOREM 2.1 (see [Ro]). *Let $P$ be a finite poset with $\hat{0}$ and $\hat{1}$. Then*

$$\sum_r (-1)^r \dim(H_r(P)) = \mu(\hat{0}, \hat{1}).$$

Let $P$ be a finite, ranked poset with $\hat{0}$ and $\hat{1}$. We say $P$ is *Cohen–Macaulay* (denoted C-M) if every interval $[x, y]$ satisfies

$$H_r([x, y]) = 0 \qquad \text{for} \quad r \neq rk(y) - rk(x) - 1. \tag{$*$}$$

The importance of the Cohen–Macaulay property comes from the following result which is an immediate corollary of Theorem 2.1.

COROLLARY 2.2. *Let $P$ be a Cohen–Macaulay poset. For every $x < y$ in $P$ we have*

$$\dim(H_r(x, y)) = \begin{cases} (-1)^{rk(y) - rk(x) - 1} \mu(x, y) & \text{if } \; r = rk(y) - rk(x) - 1 \\ 0 & \text{if } \; r \neq rk(y) - rk(x) - 1 \end{cases}$$

See [BGS] for more on poset homology and the Cohen–Macaulay property.

THEOREM 2.3. *For each $n$ and $k$, the poset $\mathscr{L}_n^{(k)}$ is Cohen–Macaulay.*

*Proof.* We prove this by induction on $n$. Let $[x, y]$ be an interval in $\mathscr{L}_n^{(k)}$. We consider several cases:

*Case* 1. $x, y \in \mathscr{T}_n^{(k)}$.

In this case, $[x, y]$ is isomorphic to the Boolean algebra on the collection of edges that are contracted to get $x$ from $y$. It is well-known that the Boolean algebras are C-M hence $[x, y]$ satisfies ($*$).

*Case* 2. $\hat{0} < x < y = \hat{1}$.

*Proof.* Let $ka_1+2, ka_2+2, ..., ka_r+2$ be the degree of the internal nodes of $x$. It is straightforward to check that the interval $[x, \hat{1}]$ is isomorphic to the direct product

$$[x, \hat{1}] \cong \mathscr{L}_{a_1}^{(k)} \times \mathscr{L}_{a_2}^{(k)} \times \cdots \times \mathscr{L}_{a_r}^{(k)} \cup \{\hat{1}\}.$$

The posets $\mathscr{L}_{a_j}^{(k)}$ are C-M by our induction hypothesis and so ( $*$ ) follows in this case by the Kunneth formulas.

*Case* 3. $[x, y] = [\hat{0}, \hat{1}]$.

This case is proved by A. Robinson in [AR] (my thanks to J.-L. Loday for pointing out this equivalence.)

This completes the result. ∎

It would be interesting to have a purely combinatorial proof of Theorem 2.3 (for example, one involving labellings) that does not depend on the topological arguments needed in Case 3.

Let $G$ be a finite group of automorphisms of $P$. Then $g \circ \partial_r = \partial_r \circ g$ for all $g \in G$ and all $r$ so $G$ acts on the kernel of $\partial_r$ and the image of $\partial_{r+1}$. This gives us a graded action of $G$ on the homology of $P$ (see Stanley [S] for a more extended discussion of group actions on poset homology).

The following generalization of Theorem 2.1 is proved in [S].

THEOREM 2.4 (The Lefschetz Fixed-Point Theorem for Posets). *Let $P$ be a finite poset with $\hat{0}, \hat{1}$ and let $G$ be a group of automorphisms of $P$. For each $r$, let $\beta_r$ denote the character of $G$ acting on $H_r(P)$. Then for each $\sigma \in G$,*

$$\sum_r (-1)^r \beta_r(\sigma) = \mu_\sigma(\hat{0}, \hat{1})$$

*where $\mu_\sigma$ denotes the Möbius function in the poset of elements in $P$ fixed by $\sigma$.*

Note that if $P$ is C-M of rank $n$ then Theorem 2.4 tells us that the character value $\beta_{n-1}(\sigma)$ of $\sigma \in G$ acting on the unique non-vanishing homology of $P$ is given by:

$$\beta_{n-1}(\sigma) = (-1)^{n-1} \mu_\sigma(\hat{0}, \hat{1}). \tag{2.5}$$

## 2B. *Cycle Indices and the Plethysm Ring*

The proof of our main result will involve some intricate calculations in a certain ring, called the plethysm ring. The elements of the plethysm ring are sequences of virtual characters of symmetric groups. The actual computations will be done by means of generating functions for elements in the plethysm ring. We will begin by defining these generating functions (which are called "cycle indices") for sequences of class functions.

Let $X_N$ be a class function on the symmetric group $S_N$. Define the *cycle index* of $X_N$ by

$$Z(X_N) = \frac{1}{N!} \sum_{\sigma \in S_N} X_N(\sigma)(x_1^{j_1(\sigma)} x_2^{j_2(\sigma)} \cdots x_N^{j_N(\sigma)}), \tag{2.6}$$

where $j_i(\sigma)$ is the number of $i$-cycles in the disjoint cycle decomposition of $\sigma$. Since two permutations in $S_N$ are conjugate if and only if they have the same cycle structure, $Z(X_N)$ completely determines the class function $X_N$.

DEFINITION 2.7. Let $X_N$ be a virtual character of $S_N$ and $\Psi_M$ a virtual character of $S_M$. Define $X_N[\Psi_M]$ to be the virtual character of $S_{NM}$ whose cycle index is given by

$$Z(X_N[\Psi_M]) = Z(X_N)[x_i \leftarrow Z(\Psi_M)[x_j \leftarrow x_{ij}]].$$

Here the notation on the right-hand side means to substitute for each $x_i$ in $Z(X_N)$, a copy of $Z(\Psi_M)$ in which each variable $x_j$ has been replaced by $x_{ij}$.

EXAMPLE 2.8. Let $N=3$ and let $X_N$ be the sign character of $S_3$. Thus $Z(X_3) = \frac{1}{6}(x_1^3 - 3x_1x_2 + 2x_3)$. Let $M=2$ and let $\Psi_2$ be the trivial character of $S_2$. Thus $Z(\Psi_2) = \frac{1}{2}(x_1^2 + x_2)$. Then

$$\begin{aligned} Z(X_3[\Psi_2]) &= \tfrac{1}{6}((\tfrac{1}{2}(x_1^2+x_2))^3 - 3(\tfrac{1}{2}(x_1^2+x_2))(\tfrac{1}{2}(x_2^2+x_4)) + 2(\tfrac{1}{2}(x_3^2+x_6)) \\ &= \tfrac{1}{720}(15x_1^6 + 45x_1^4x_2 - 45x_1^2x_2^2 - 75x_2^3 - 90x_1^2x_4 \\ &\quad - 90x_2x_4 + 120x_3^2 + 120x_6). \end{aligned}$$

So $X_3[\Psi_2]$ is the character of $S_6$ whose values are given in the chart below:

| Cycle lengths of $\sigma$ | $X_3[\Psi_2](\sigma)$ |
|---|---|
| $1^6$ | 15 |
| $1^42$ | 3 |
| $1^22^2$ | $-1$ |
| $2^3$ | $-5$ |
| $1^24$ | $-1$ |
| 24 | $-1$ |
| $3^2$ | 8 |
| 6 | 1 |

DEFINITION 2.9. Define the *plethysm ring* $P$ to be the collection of all sequences $X=(X_1, X_2, \ldots)$ where $X_N$ is a virtual character of $S_N$. Define $+$ and $[\ ]$ in $P$ by

$$X+\Psi := (X_1+\Psi_1, X_2+\Psi_2, \ldots)$$
$$X[\Psi] = (\Gamma_1, \Gamma_2, \ldots)$$

where

$$\Gamma_n = \sum_{r \mid n} X_r[\Psi_{n/r}].$$

It is straightforward to check that $(P, +, [\ ])$ is an associative ring with multiplicative identity $\varepsilon = (\varepsilon_1, 0, 0, 0, \ldots)$ where $\varepsilon_1$ is the trivial character of $S_1$.

DEFINITION 2.10. For $X=(X_1, X_2, \ldots)$ in $P$ define $Z(X)$ by $Z(X) = \sum_{n=1}^{\infty} Z(X_n)$. From Definition 2.7 it follows that

$$Z(X+\Psi) = Z(X) + Z(\Psi)$$

and

$$Z(X[\Psi]) = Z(X)[Z(\Psi)]$$

where $A[B]$ denotes $A[x_i \leftarrow B[x_j \leftarrow x_{ij}]]$ for $A, B$ power series in $\mathbf{Q}[[x_1, x_2, \ldots]]$.

## 2C. *Partitions with Restricted Block Size*

For each $N$ let $\tilde{\Pi}_N^{(k)}$ denote the collection of (set) partitions of $1, 2, \ldots, N$ which have the property that every block size is congruent to $1 \bmod k$. Order the partitions in $\tilde{\Pi}_N^{(k)}$ by refinement thus making it into a partially ordered set. Note that $\tilde{\Pi}_N^{(k)}$ has a unique maximal element only if $N \equiv 1 \bmod k$. Define $\Pi_N^{(k)}$ to be

$$\Pi_N^{(k)} = \begin{cases} \tilde{\Pi}_N^{(k)} & \text{if } N \equiv 1 \pmod{k} \\ \tilde{\Pi}_N^{(k)} \cup \{\hat{1}\} & \text{if } N \not\equiv 1 \pmod{k} \text{ where } \hat{1} \text{ is a new maximal element.} \end{cases}$$

Bjorner [B] proved that $\Pi_N^{(k)}$ is Cohen–Macaulay. Note that the symmetric group $S_N$ acts as a group of automorphisms of $\Pi_N^{(k)}$ hence acts on the unique non-vanishing homology group of $\Pi_N^{(k)}$. In [CHR], the authors studied the character values of this representation of $S_N$ on $H_*(\Pi_N^{(k)})$. In particular, they showed that a remarkable equation relates these character values to the character values of the trivial characters.

DEFINITION 2.11. For each $j \in \{0, 1, 2, ..., k-1\}$, let $\beta^{(k,j)}$ and $\varepsilon^{(k,j)}$ be the elements of $P$ given as follows. First of all, $\beta_N^{(k,j)}$ and $\varepsilon_N^{(k,j)}$ are 0 if $N$ is incongruent to $j \bmod k$. For $N \equiv j \bmod k$, let $\varepsilon_N^{(k,j)}$ be the trivial character of $S_N$ and let $\beta_N^{(k,j)}$ be the virtual character obtained by multiplying the character of $S_N$ on the top homology of $\Pi_N^{(k)}$ by $(-1)^{(N-j)/k}$.

The following theorem will be a crucial ingredient in the next section.

THEOREM 2.12 [CHR, Theorem 4.7]. (a) *$\beta^{(k,1)}$ and $\varepsilon^{(k,1)}$ are inverses in the pethysm ring. In cycle index terms,*

$$Z(\beta^{(k,1)})[Z(\varepsilon^{(k,1)})] = Z(\varepsilon^{(k,1)})[Z(\beta^{(k,1)})] = x_1$$

(b) *For $j \neq 1$, $\varepsilon^{(k,j)}[\beta^{(k,1)}] = -\beta^{(k,j)}$.*

2D. *Lie k-Algebras*

A classical result in Algebraic Combinatorics states that the top homology of the partition lattice $\Pi_N$ (twisted by the sign representation) is isomorphic as an $S_N$-module to the $1^N$-homogeneous piece of the free Lie algebra on $N$ generators. Hanlon and Wachs introduced the notion of a Lie $k$-algebra in order to prove a $k$-analogue of this result in which $\Pi_N$ is replaced by $\Pi_N^{(k)}$ (see [HW]). We briefly review the notion of a Lie $k$-algebra.

Let $V = V_0 \oplus V_1$ be a bigraded vector space. Make $T(V)$ into a bigraded vector space by

$$(V^{\otimes f})_\varepsilon = \bigoplus_{\substack{(\varepsilon_1, ..., \varepsilon_f) \\ \sum \varepsilon_i \equiv \varepsilon (\mathrm{mod}\ 2)}} (V_{\varepsilon_1} \otimes \cdots \otimes V_{\varepsilon_f}).$$

DEFINITION 2.13. Let $\sigma \in S_f$ and let $x_i \in V_{\varepsilon_i}$. Define the "Koszul sign," $sgn(\sigma, x)$ by

$$sgn(\sigma; x_1 \otimes \cdots \otimes x_f) = \prod_{\substack{i<j \\ \sigma_i > \sigma_j}} (-1)^{\varepsilon_i \varepsilon_j}.$$

For convenience we usually denote $sgn(\sigma; x_1 \otimes \cdots \otimes x_f)$ by $sgn(\sigma; x)$. Note that $sgn(\sigma; x)$ is the usual $sgn(\sigma)$ if all $\varepsilon_i's$ are 1.

DEFINITION 2.14. A *Lie k-bracket* is a multilinear map [ ] from $V^{\otimes (k+1)}$ to $V$ which satisfies:

(a) $V_\varepsilon^{\otimes (k+1)} \to V_{\varepsilon+1}$

(b) [ ] alternates with respect to the grading, i.e., if $w_i \in V_{\varepsilon_i}$ and $\sigma \in S_{k+1}$. Then

$$[w_{\sigma^{-1}1}, ..., w_{\sigma-1(k+1)}] = sgn(\sigma; w)[w_1, ..., w_{k+1}].$$

(c) If $w_i \in V_{\varepsilon_i}$ for $i = 1, 2, ..., 2k+1$. Then

$$\sum_{\sigma \in S_{2k+1}} sgn(\sigma; w)[w_{\sigma^{-1}1}, ..., w_{\sigma^{-1}k}, [w_{\sigma^{-1}(k+1)}, ..., w_{\sigma^{-1}(2k+1)}]] = 0.$$

A *Lie k-Algebra* is a bigraded vector space $V$ endowed with a Lie $k$-bracket.

Let $F_N^{(k)}$ denote the free Lie $k$-algebra with $N$ odd generators. So $F_N^{(k)}$ is the smallest vector space which contains the generators $\gamma_1, ..., \gamma_N$, and is closed under $k$-bracketting (see [HW] Section 2 for a more precise construction of $F_N^{(k)}$). $F_N^{(k)}$ is $\mathbf{N}^N$-graded. For $\alpha \in \mathbf{N}^N$, the $\alpha$-graded piece of $F_N^{(k)}$ is the span of all brackets which contain $\alpha_i$ occurrences of the generator $\gamma_i$. We denote the $\alpha$-graded piece of $F_N^{(k)}$ by $F_N^{(k)}[\alpha]$.

DEFINITION 2.15. For $N \equiv 1 (\mathrm{mod}\, k)$ let $Lie_N^{(k)}$ denote the character of $S_N$ acting on the $1^N$-homogeneous piece of $F_N^{(k)}$.

In the next section we will need an appropriate notion of $Lie_N^{(k)}$ for all $N$. Unfortunately the $1^N$ homogeneous piece of $F_N^{(k)}$ is zero for $N \not\equiv 1 (\mathrm{mod}\, k)$. We will employ an alternate definition of $Lie_N^{(k)}$ which is non-zero for all $N$ and which equals the character of the $S_N$ action on $F_N^{(k)}[1^N]$ when $N \equiv 1 (\mathrm{mod}\, k)$.

By the definition of Lie $k$-algebra, there is a notion of homology for $F_N^{(k)}$ defined in terms of a Koszul complex $C_*(F_N^{(k)})$:

$$\cdots \to \Lambda^d F_N^{(k)} \to \Lambda^{d-k} F_N^{(k)} \to \Lambda^{d-2k} F_N^{(k)} \to \cdots. \tag{2.16}$$

The $\mathbf{N}^N$-grading on $F_N^{(k)}$ extends to an $\mathbf{N}^N$-grading on the complex (2.16) in the usual way. For $\alpha \in \mathbf{N}^N$ let $\Lambda^{d,\alpha} F_N^{(k)}$ denote the $\alpha$-graded piece in $\Lambda^d F_N^{(k)}$.

Define $\tilde{C}_*(F_N^{(k)})$ to be the complex (2.16) truncated to degrees $d > 1$. Note that

$$\Lambda^{1, 1^N}(F_N^{(k)}) = 0 \qquad \text{unless} \quad N \equiv 1 (\mathrm{mod}\, k).$$

So the $1^N$-graded piece of $\tilde{C}_*(F_N^{(k)})$ equals the $1^N$-graded piece of $C_*(F_N^{(k)})$ unless $N \equiv 1 (\mathrm{mod}\, k)$.

DEFINITION 2.17. Define $Lie_N^{(k)}$ to be the character of $S_N$ acting on the $1^N$-homogeneous piece of the homology of $\tilde{C}_*(F_N^{(k)})$.

The following result can be obtained by combining Theorems 3.11 and 4.13 from [HW].

THEOREM 2.18 (Hanlon–Wachs). *$Lie_N^{(k)}$ equals the character of $S_N$ acting on the top homology of $\Pi_N^{(k)}$.*

## 3. THE $S_{nk+2}$ ACTION ON $H_*(\mathscr{L}_n^{(k)})$

Recall from Section 1 the definition of the poset $\mathscr{L}_N^{(k)}$ consisting of all homeomorphically irreducible $k$-trees with $nk+2$ labelled leaves together with an appended maximal element. By Theorem 2.3, $\mathscr{L}_N^{(k)}$ is a Cohen–Macaulay poset of rank $n$. The purpose of this section is to prove Theorem 1.1, the main result of this paper, which identifies the action of $S_{nk+2}$ on the unique non-vanishing homology of $\mathscr{L}_n^{(k)}$.

The proof will be via a cycle index computation. We will compute the cycle indices of the two sides of Theorem 1.1 and show that they are equal.

DEFINITION. Let $\lambda_{nk+2}^{(k)}$ denote the character of $S_{nk+2}$ acting on the top homology of $\mathscr{L}_n^{(k)}$.

By Theorem 2.4 we can compute the value of $\lambda_{nk+2}^{(k)}$ on a permutation $\sigma$ using the Möbius function $\mu_\sigma$ of the fixed-point poset $(\mathscr{L}_n^{(k)})^\sigma = \{x \in \mathscr{L}_n^{(k)}: \sigma(x) = x\}$. The formula is

$$\lambda_{nk+2}^{(k)}(\sigma) = (-1)^n \mu_\sigma(\hat{0}, \hat{1}). \tag{3.1}$$

For $x$ a tree fixed by $\sigma$, the interval $[\hat{0}, x]$ in $(\mathscr{L}_n^{(k)})^\sigma$ is isomorphic to the Boolean algebra on the set of cycles of $\sigma$ acting on the internal edges of $x$. Combining (3.1) with that observation and the usual recursion for the Möbius function we have:

$$\lambda_{nk+2}^{(k)}(\sigma) = (-1)^{n-1} \Sigma_\tau (-1)^{I(\tau;\sigma)} \tag{3.2}$$

where the sum is over all trees $\tau \in \mathscr{T}_n^{(k)}$ which are fixed by $\sigma$ and where $I(\tau; \sigma)$ is the number of cycles of $\sigma$ acting on the set of internal edges of $\tau$.

*Note.* Because (3.2) fails for $n=0$, our cycle index sum computation will be incorrect in degree 2. However, the reader can easily check that the main result (Theorem 1.1) holds in degree 2. So we will proceed with the computation and ignore errors in the final formula for $Z(\lambda_2^{(k)})$.

We will need a generalized form of the classical statement of Burnside's Lemma. Let $G$ be a finite group acting on a finite set $S$, and let Fix denote the collection of pairs $(g, s) \in G \times S$ such that $g(s) = s$. Let $\gamma$ be a function from Fix to a commutative $Q$-algebra $R$ which satisfies $\gamma(hgh^{-1}, hs) = \gamma(g, s)$ for all $h \in G$, $(g, s) \in Fix$. Then for $s \in S$ define $Z_\gamma(s)$ by

$$Z_\gamma(s) = \frac{1}{|G_s|} \sum_{g \in G_s} \gamma(g, s),$$

where $G_s$ denotes the stabilizer of $s$. It is easy to check that $Z_\gamma$ is a $G$-invariant function on $S$, namely $Z_\gamma(gs) = Z_\gamma(s)$ for $g \in G$. So for $\mathcal{O}$ an orbit of the action of $G$ on $S$, define $Z_\gamma(\mathcal{O})$ to be $Z_\gamma(s)$ for any $s \in \mathcal{O}$. The next result tells us how to sum the values of $Z_\gamma(\mathcal{O})$ over the orbits $\mathcal{O}$.

THEOREM 3.3 [RR]. *With notation as above,*

$$\sum_{\mathcal{O}} Z_\gamma(\mathcal{O}) = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{s \in S \\ gs = s}} \gamma(g, s).$$

DEFINITION 3.4. Let $h^{(k)}$ and $H^{(k)}$ denote the following collections of unlabelled trees:

$h_n^{(k)}$ = trees with $nk + 2$ leaves in which every internal node has degree of the form $mk + 2$ for $m \geqslant 1$.

$H_n^{(k)}$ = trees rooted at a leaf whose unrooted versions lie in $h_n^{(k)}$.

For $t$ in $h_n^{(k)}$, define $Z_\gamma(t)$ by the following procedure:

*Step* 1. Pick a labelled tree $\tau \in \mathcal{T}_n^{(k)}$ which is equal to $t$ when the labels are removed.

*Step* 2. Let $A(\tau)$ denote the automorphism group of $\tau$. For $\sigma \in A(\tau)$ define

$$\gamma(\sigma, \tau) = (-1)^{E(\tau;\, \sigma)} Z(\sigma). \tag{3.5}$$

where $E(\tau; \sigma)$ is the number of cycles of $\sigma$ acting on the edge set of $\tau$ and $Z_\sigma$ denotes the cycle indicator of $\sigma$ acting on the vertices of $\tau$. Note that $\gamma(\pi\sigma\pi^{-1}, \pi\tau) = \gamma((\sigma, \tau)$.

*Step* 3. Define $Z_\gamma(t) = (1/A(\tau)) \sum_{\sigma \in A(\tau)} \gamma(\sigma, \tau)$. This is well-defined by the fact that an unlabelled graph in $h_n^{(k)}$ is simply an orbit of labelled graphs in $T_n^{(k)}$ under the action $S_{nk+2}$.

THEOREM 3.6. *Let $w$ be a primitive $(2k)$th root of unity. Then*

$$w^2 Z(\lambda^{(k)}) = -\left(\sum_n \sum_{t \in h_n^{(k)}} Z_\gamma(t)\right)[x_i \leftarrow -w^i x_i] + (wx_1)^2$$

*where the notation $GLOB[A \leftarrow B]$ denotes the result of substituting $B$ for every occurrence of $A$ in $GLOB$.*

*Proof.* Suppose that $\tau \in \mathscr{T}_n^{(k)}$ and $\sigma \in A(\tau)$ and suppose $\tau$ has more than two vertices. Then

$$\begin{aligned}\gamma(\tau, \sigma)[x_i \leftarrow -w^i x_i] &= w^{nk+2} \gamma(\tau, \sigma)[x_i \leftarrow -x_i] \\ &= (-1)^n w^2 (-1)^{E(\tau, \sigma)} Z(\sigma)[x_i \leftarrow -x_i] \\ &= (-1)^n w^2 (-1)^{I(\tau, \sigma)} Z(\sigma).\end{aligned}$$

The last step uses the fact that the number of cycles of $\sigma$ acting on the leaves of $\tau$ equals the number of cycles of $\sigma$ acting on the non-interior edges of $\tau$ (except in the case when $\tau$ has two vertices and $\sigma = (1, 2)$). Also,

$$\left(\sum_n \sum_{t \in h_n^{(k)}} Z_\gamma(t)\right) = \sum_n \frac{1}{(nk+2)!} \sum_{\sigma \in S_{nk+2}} \sum_{\tau \in (\mathscr{T}_n^{(k)})_\sigma} \gamma(\tau, \sigma)$$

by Theorem 3.3.

So

$$\begin{aligned}-&\left(\sum_n \sum_{t \in h_n^{(k)}} Z_\gamma(t)\right)[x_i \leftarrow -w^i x_i] \\ &= w^2 \sum_{n \geqslant 1} \frac{(-1)^n}{(nk+2)!} \sum_{\sigma \in S_{nk+2}} \sum_{\tau \in (\mathscr{T}_n^{(k)})_\sigma} (-1)^{I(\tau; \sigma)} Z(\sigma) - \frac{w^2}{2}(x_1^2 - x_2) \\ &= w^2 \sum_{n \geqslant 0} \sum_{\sigma \in S_{nk+2}} \beta_{nk+2}^{(k)}(\sigma) Z(\sigma) - (wx_1)^2. \quad \blacksquare\end{aligned}$$

In view of Theorem 3.6, $Z(\lambda^{(k)})$ can be computed from a sufficiently explicit expression for

$$Z_\gamma(h^{(k)}) = \sum_n \sum_{t \in h_n^{(k)}} Z_\gamma(t)$$

We will compute $Z(h^{(k)})$ using a classic enumerative technique involving dissimilarity characteristics due to Richard Otter (see [O]).

For $T \in H_n^{(k)}$ define $Z_\gamma(T)$ in the same way that $Z_\gamma(t)$ was defined for unrooted trees $t \in h_n^{(k)}$ except that the terms in $Z_\gamma(T)$ will not include a factor of $x_1$ for the rooted point. Define

$$Z_\gamma(H^{(k)}) = \sum_n \sum_{T \in H_n^{(k)}} Z_\gamma(T).$$

Our first step will be to compute $Z_\gamma(H^{(k)})$.

THEOREM 3.7. $Z_\gamma(H^{(k)}) = -x_1 - \sum_{n=1}^{\infty} Z(\varepsilon_{nk+1})[Z_\gamma(H^{(k)})]$, *where* $\varepsilon_N$ *denotes the trivial character of* $S_N$ (*so* $Z(\varepsilon_N) = (1/N!) \sum_{\sigma \in S_N} Z(\sigma)$).

*Proof.* There is one planted tree in which the root point is adjacent to another leaf, namely

$T =$ 

For this $T$ we have $Z_\gamma(T) = -x_1$ (remember that we are not indicating a 1-cycle for the rooted point in $Z_\gamma(T)$).

Otherwise the root in a planted tree must be adjacent to an interior point of degree $nk+1$ for some $n \geqslant 1$. Such trees can be identified with orbits of functions from $\{1, 2, ..., nk+1\}$ to $H^{(k)}$ under the action of $S_{nk+1}$. By the Composition Theorem ([HP, pg. 182] or [RR]) the sum of $Z_\gamma(t)$ over all such $T$ is

$$-Z(\varepsilon_{nk+1})[Z_\gamma(H^{(k)})].$$

Here the minus sign accounts for the fact that the edge emanating from the root is in a 1-cycle under any automorphism. This 1-cycle must receive a minus sign in $\gamma(\tau, \sigma)$ which is not accounted for in $Z(\varepsilon_{nk+1})[Z_\gamma(H^{(k)})]$. ∎

COROLLARY 3.8. *For each* $N$, *let* $Lie_N^{(k)}$ *denote the character of the representation of* $S_N$ *on the* $1^N$*-graded piece of the free Lie algebra with* $N$ *odd generators. Then*

$$Z_\gamma(H^{(k)})[x_i \leftarrow -w^i x_i] = w \sum_n \frac{1}{(nk+1)!} \sum_{\sigma \in S_{nk+1}} Lie_{nk+1}^{(k)}(\sigma)\, Z(\sigma).$$

*Proof.* We begin with a lemma.

LEMMA 3.9. *Suppose* $A$, $G1$, $G2 \in \mathbf{C}[[x_1, x_2, \ldots]]$ *satisfy*

$$x_1 = A[G1] = (-A)[G2].$$

*Then* $G2 = G1[x_i \leftarrow -x_i]$.

*Proof of Lemma.* Since $U[V]$ is an associative product for which $x_1$ is the identity, $x_1 = A[G1]$ implies that $x_1 = G1[A]$. Also note that $x_1 = (-A)[G2]$ is equivalent to $-x_1 = A[G2]$. Combining these observations we have

$$\begin{aligned} G2 &= x_1[G2] = (G1[A])[G2] = G1[A[G2]] \\ &= G1[-x_1] = G1[x_i \leftarrow -x_i]. \end{aligned}$$

This proves Lemma 3.9.

Next we combine Theorem 2.12(a) with Theorem 2.18 from [HW] to obtain that

$$x_1 = \left( \sum_{n=0}^{\infty} Z(\varepsilon_{nk+1}) \right) \left[ \sum_{n=0}^{\infty} (-1)^n Z(Lie_{nk+1}^{(k)}) \right]. \tag{3.10}$$

It follows from 3.10, Lemma 3.9 and Theorem 3.7 that

$$\sum_{n=0}^{\infty} (-1)^n Z(Lie_{nk+1}^{(k)})[x_i \leftarrow -x_i] = Z_\gamma(H^{(k)}). \tag{3.11}$$

Substituting $x_i \leftarrow -w^i x_i$ on both sides of 3.11 yields

$$\sum_{n=0}^{\infty} Z(Lie_{nk+1}^{(k)}) = Z_\gamma(H^{(k)})[x_i \leftarrow -w^i x_i]. \quad \blacksquare$$

We need to make some comments on trees. Let $t$ be a tree. For each vertex $v$ in $t$, let $d(v)$ denote the length of the longest path in $t$ which begins at $v$. It is well-known that $d$ is minimized on either a single point $v$ or on a pair of adjacent points $u$, $v$. In the former case we call $v$ the *center* of $t$ and in the latter case we call $u$, $v$ and the edge joining them the *bicenter* of $t$. The center or bicenter of $t$ is invariant under $A(t)$. A *symmetry edge* of $t$ is an edge $e$ whose endpoints are interchanged by some automorphism of $t$. It is not hard to check that if $e$ is a symmetry edge of $t$ then $t$ has a bicenter consisting of $e$ and its endpoints.

A vertex-rooting of $t$ is a choice of distinguished vertex $v$ in $t$. Two vertex-rootings $T_1$, $T_2$ of $t$ are identical if there is an automorphism of $t$ which maps the root of $T_1$ to the root of $T_2$. So there is one vertex-rooting of $t$ for every orbit of $A(t)$ on the vertices of $t$. The automorphism group $A(T)$ of $t$ rooted at a vertex $v$ is the subgroup of $A(t)$ which fixes $v$.

In a similar way, we can define an edge-rooted tree $T$. Again, an edge-rooting of $t$ consists of $t$ with a certain edge distinguished. Two edge-rootings are identical if there is an automorphism which maps the rooted edge of one to the rooted edge of the other. But there is a subtlety that arises in the case where we have rooted $t$ at a symmetry edge. We say $T_e$ is a *strong edge-rooting of t at e* if $A(T_e)$ consists of those automorphisms of $t$ which fix both endpoints of $e$. We say $T_e$ is a *weak edge-rooting of t at e* if $A(T_e)$ consists of those automorphisms of $t$ which fix $e$ (included in $A(T_e)$ are any automorphisms which interchange the endpoints of $e$). We move on now to the computation of $Z_\gamma(h^{(k)})$. To fully justify the dissimilarity characteristic method we will need one more lemma.

LEMMA 3.12. *Let t be a tree, let $V_t$ be the collection of point-rootings of t, let $E_t$ be the collection of strong edge-rootings of t and let $S_t$ be the collection of weak rootings of t at a symmetry edge. Let $\gamma$ be any class function defined on pairs $(\tau, \sigma)$ where $\tau$ is a rooted tree and $\sigma \in A(\tau)$. For $T \in V_t$, $T \in E_t$ or $T = t$ define $Z_\gamma(T)$ as above—choose a labelled version $\tau$ of T and define $Z_\gamma(T) = (1/|A(\tau)|) \sum_{\sigma \in A(\tau)} \gamma(\tau, \sigma)$. Then*

$$Z_\gamma(t) = \left(\sum_{T \in V_t} Z_\gamma(T)\right) - \left(\sum_{T \in E_t} Z_\gamma(T)\right) + \left(\sum_{T \in S_t} Z_\gamma(T)\right).$$

*Proof.* Let $\tau$ be a labelled version of $t$. Let $C$ be the center or bi-center of $\tau$ (whichever is applicable). Let $e$ be an edge of $\tau$ not in $C$ and let $v$ be the vertex incident to $e$ which is furthest from $C$. Also let $\tau(e)$ and $\tau(v)$ denote the rootings of $\tau$ at $e$ and $v$ respectively. Then $A(\tau(e)) = A(\tau(v))$ and so

$$Z_\gamma(\tau(v)) - Z_\gamma(\tau(e)) = 0. \tag{3.13}$$

It follows from (3.13) that

$$\sum_{T \in V_t} Z_\gamma(T) - \sum_{T \in E_t} Z_\gamma(T) + \sum_{T \in S_t} Z_\gamma(T)$$

is equal to

$$\sum_{T \in V_t^c} Z_\gamma(T) - \sum_{T \in E_t^c} Z_\gamma(T) + \sum_{T \in S_t} Z_\gamma(T).$$

where $V_t^c$ is the subset of $V_t$ consisting of those rooted trees where the rooted point is either in a center or bicenter and $E_t^c$ is the subset of $E_t$ consisting of those edge-rooted trees where the rooted edge is in a bicenter. We consider three cases:

*Case* 1. $t$ has a center $v$. In this case $E_t^c$ and $S_t$ are empty and $V_t^c$ consists of just one tree $T$ which has $v$ rooted. Since $v$ is the center of $t$, $A(T)=A(t)$ and so

$$Z_\gamma(T)=Z_\gamma(t).$$

Lemma 3.12 follows in this case.

*Case* 2. $t$ has a bicenter consisting of vertices $u, v$ joined by an edge $e$ and $e$ is not a symmetry edge. In this case $S_t$ is empty and $E_t^c$ consists of one tree $T_e$ rooted at the edge $e$. Since $e$ is not a symmetry edge, $A(T_e)=A(t)$ so

$$Z_\gamma(T_e)=Z_\gamma(t).$$

The set $V_t^c$ consists of two trees $T_u$, $T_v$ where $T_x$ has $x$ as its rooted vertex. Since $u$ and $v$ are the vertices in the non-symmetric bicenter of $t$, $A(T_u)=A(T_v)=A(t)$ and

$$Z_\gamma(T_u)=Z_\gamma(T_v)=Z_\gamma(t).$$

So

$$\sum_{T\in V_t^c} Z_\gamma(T)-\sum_{T\in E_t^c} Z_\gamma(T)=Z_\gamma(T_u)+Z_\gamma(T_v)-Z_\gamma(T_e)=Z_\gamma(t).$$

This proves the lemma in this case.

*Case* 3. $t$ has a bicenter consisting of vertices $u, v$ joined by an edge $e$ where $e$ is a symmetry edge. In this case $V_t^c$ consists of just one tree $T_v$ which is rooted at $v$. This is because $u$ and $v$ are in the same orbit of $A(t)$ so the rootings at $u$ and at $v$ are equivalent. The set $E_t^c$ consists of a single graph $T_e$ which has $t$ strongly rooted at the edge $e$. Because $e$ is strongly rooted at $e$, $A(T_e)=A(T_v)(\neq A(t))$ and so

$$Z_\gamma(T_v)-Z_\gamma(T_e)=0.$$

The set $S_t$ contains one tree $W_e$ which consists of $t$ with $e$ weakly rooted. Since $e$ is in the bicenter of $t$, $A(W_e)=A(t)$ so

$$Z_\gamma(W_e)=Z_\gamma(t).$$

Thus

$$\sum_{T\in V_t^c} Z_\gamma(T)-\sum_{T\in E_t^c} Z_\gamma(T)+\sum_{T\in S_t} Z_\gamma(T)=Z_\gamma(T_v)-Z_\gamma(T_e)+Z_\gamma(W_e)=Z(t)$$

This proves Lemma 3.12 in Case 3 and completes the argument. ∎

We are now going to apply Lemma 3.12 to compute $Z_\gamma(h^{(k)})$. We will sum both sides of the equality in Lemma 1.14 over all $t \in h^{(k)}$. On the left-hand side we obtain $Z_\gamma(h^{(k)})$. On the right-hand side we obtain three terms which we will compute separately.

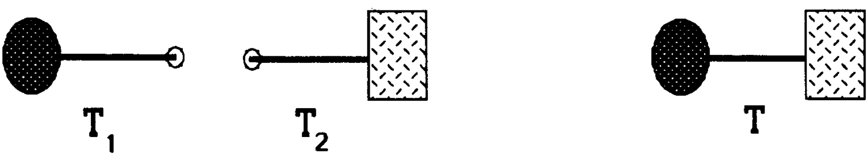
LEMMA 3.14.

$$\sum_{t \in h^{(k)}} \sum_{T \in V_t} Z_\gamma(T) = x_1 Z_\gamma(H^{(k)}) + \sum_{n=1}^{\infty} Z(\varepsilon_{nk+2})[Z_\gamma(H^{(k)})].$$

*Proof.* The left-hand side sums $Z_\gamma(T)$ over all non-isomorphic unlabelled vertex-rooted trees whose unrooted versions lie in $h^{(k)}$. Such trees are either rooted at a leaf or rooted at an internal node. The term $x_1 Z_\gamma(H^{(k)})$ accounts for the former set and the sum on the right accounts for the latter set. ∎

LEMMA 3.15. $\sum_{t \in h^{(k)}} \sum_{T \in E_t} Z_\gamma(T) = \frac{-1}{2} Z_\gamma(H^{(k)})^2 - \frac{1}{2} \sum_{T_1 \in H^{(k)}} Z_\gamma(T_1)^2.$

*Proof.* Following Harary and Prins [HPr], we will construct all strongly edge-rooted trees $T$ as follows. Pick a pair $T_1, T_2 \in H^{(k)}$ and identify the edge from the rooted leaf in $T_1$ with the edge from the rooted leave in $T_2$.




Since $T$ is strongly edge-rooted at $e$,

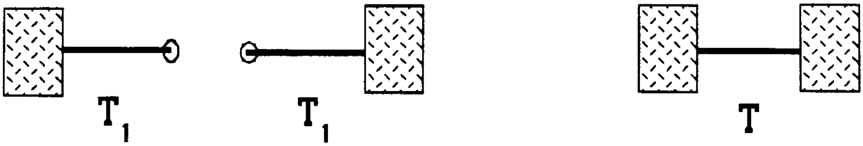
$$Z_\gamma(T) = Z_\gamma(T_1)\, Z_\gamma(T_2).$$

This procedure constructs each rooted tree $T$ twice if $T_1$ and $T_2$ are different but only once if $T_1 = T_2$ (i.e., if $e$ is a symmetry edge). The result follows.

One other comment needs to be made to account for the minus sign. The edges $e_1$ and $e_2$ that are identified to become $e$, each contribute a minus sign to $Z_\gamma(T_1)\, Z_\gamma(T_2)$. But the edge $e$ should contribute only one minus sign which explains why we include the minus signs on the right-hand side of Lemma 3.15. ∎

LEMMA 3.16. $\sum_{t \in h^{(k)}} \sum_{T \in S_t} Z_\gamma(T) = \frac{1}{2}(-\sum_{T_1 \in H^{(k)}} Z_\gamma(T_1)^2 + x_2[Z_\gamma(H^{(k)})]).$

*Proof.* We construct all $T \in S_t$ by taking two copies of a tree $T_1 \in H^{(k)}$ and identifying their edges incident to the rooted leaves as above.



The automorphism group of $T$ is the wreath product of $A(T_1)$ over $S_2$. So

$$Z_\gamma(T) = \tfrac{1}{2}(-x_1^2 + x_2)[Z_\gamma(T_1)].$$

The sign in $-x_1^2$ again takes into account the sign-weighting of the rooted edge.

The result follows by summing over $T_1 \in H^{(k)}$. ∎

THEOREM 3.17 (Dissimilarity Characteristic Equation for $Z(h^{(k)})$).

$$Z_\gamma(h^{(k)}) = x_1 Z_\gamma(H^{(k)}) + \sum_{N \equiv 2(k)} Z(\varepsilon_N)[Z_\gamma(H^{(k)})].$$

*Proof.* Summing Lemma 3.16 over all $t \in h^{(k)}$ and using Lemmas 3.14, 3.15, and 3.16 we have:

$$\begin{aligned} Z_\gamma(h^{(k)}) &= x_1 Z_\gamma(H^{(k)}) + \sum_{n=1}^{\infty} Z(\varepsilon_{nk+2})[Z_\gamma(H^{(k)})] \\ &\quad + \frac{x_1^2}{2}[Z_\gamma(H^{(k)})] + \frac{1}{2} \sum_{T_1 \in H^{(k)}} Z_\gamma(T_1)^2 \\ &\quad - \frac{1}{2}\left(\sum_{T_1 \in H^{(k)}} Z_\gamma(T_1)^2\right) + \frac{x_2}{2}[Z_\gamma(H^{(k)})] \\ &= x_1 Z_\gamma(H^{(k)}) + \sum_{n=0}^{\infty} Z(\varepsilon_{nk+2})[Z_\gamma(H^{(k)})]. \quad ∎ \end{aligned}$$

We can now prove the main result of this section.

THEOREM 3.18. *For all n,*

$$\lambda_{nk+2}^{(k)} = ind_{S_{nk+1}}^{S_{nk+2}}(Lie_{nk+1}^{(k)})/Lie_{nk+2}^{(k)}.$$

*Proof.* From Theorem 3.6 and Theorem 3.17 we have

$$Z(\lambda^{(k)}) = -w^{-2} Z_\gamma(h^{(k)})[x_i \leftarrow -w^i x_i] + w^{-2}(wx_1)^2$$

$$= -w^{-2}\Bigg(-wx_1(Z_\gamma(H^{(k)})[x_i \leftarrow -w^i x_i])$$

$$+\Bigg(\sum_{N\equiv 2(k)} Z(\varepsilon_N)[Z_\gamma(H^{(k)})]\Bigg)[x_i \leftarrow w^i x_i]\Bigg) + x_1^2.$$

By Theorem 3.7 and Theorem 2.12(a), it follows that

$$Z_\gamma(H^{(k)}) = \sum_{N\equiv 1(k)} \frac{1}{N!} \sum_{\sigma\in S_N} \mu_\sigma(\hat{0}, \hat{1})\, Z(\sigma)[x_i \leftarrow -x_i]$$

where $\mu_\sigma$ is the Möbius function of the subposet of $\Pi_N^{(1,k)}$ fixed by $\sigma$. So by Theorem 2.12(b)

$$\sum_{N\equiv 2(k)} Z(\varepsilon_N)[Z_\gamma(H^{(k)})] = -\sum_{N\equiv 2(k)} \frac{1}{N!} \sum_{\sigma\in S_N} \mu_\sigma(\hat{0}, \hat{1})\, Z(\sigma)[x_i \leftarrow -x_i].$$

Now by Theorem 3.11 of [HW] we have

$$\Bigg(\sum_{N\equiv 2(k)} Z(\varepsilon_N)[Z_\gamma(H^{(k)})]\Bigg)[x_i \leftarrow -w^i x_i] = -w^2 \sum_{N\equiv 2(k)} Z(Lie_N^{(k)}).$$

Combining this with Corollary 3.8 we have

$$Z(\gamma^{(k)}) = \sum_{N\equiv 2(k)} Z((Lie_{N-1} \uparrow_{S_{N-1}}^{S_N})/Lie_N) + x_1^2.$$

This proves Theorem 3.18 and the main result.

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